Clay Math – generalized geometry workshop

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McKay correspondence in the context of Hopf actions on AS regular algs

joint w/ Kenneth Chan, Ellen Kirkman, James Zhang

/ $\mathbb{C}$

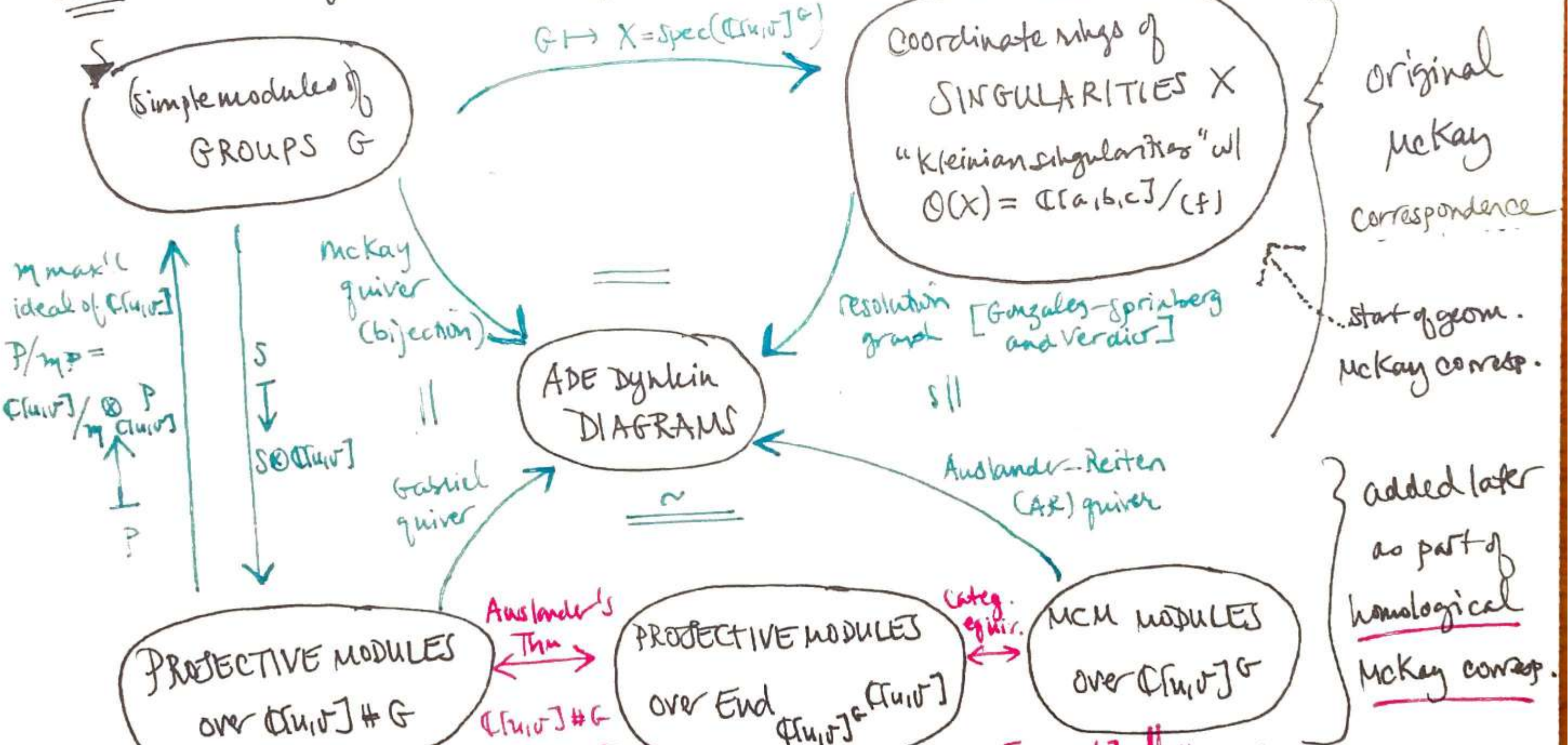
Refs: I. 1607.06977 (case by case proofs)

II. 1610.01220 (adaptation of 'classical' homological arguments)

An incomplete picture

A reminder of the McKay correspondence in the 'classical' setting....

Start with G = finite subgroup of $SL_2(\mathbb{C})$ that acts linearly & faithfully on $\mathbb{C}[u,v]$



B (comm., local) algebra of finite Krull dim

A algebra $\in$ mod B

add$_B A$ = full subcateg. of mod B of all direct summands of finite $\oplus$ of copies of A

prmod A = full subcateg. of mod B of all projective A-modules

MCM-B = full subcateg. of mod B consisting of M w/ depth $M \overset{CM}{=} \dim M \overset{max'l}{=} \dim B$

Also related to... ... out of scope of talk

- SINGULARITY CATEGORIES — derived McKay ...
- PREPROJECTIVE ALGEBRAS — deformed McKay ...
- MATRIX FACTORIZATIONS — more on MCMs ...
- CLUSTER CATEGORIES — more MCMs / tilting thy ...

Nice Ref: Leuschke–Wiegand's book on CM mods

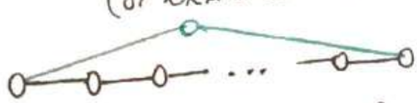
A reminder of the DIAGRAMS $\quad G \leq SL_2(\mathbb{C})$ with fundamental rep. V $(= \mathbb{C}u \oplus \mathbb{C}v)$

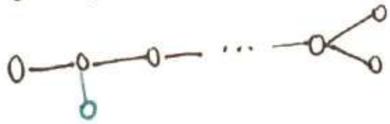
DIAGRAM	Vertices	arrows/edges
McKay quiver	corresponding to the set of non-isom. simple modules of G $V_0, V_1, \dots, V_d$ (V_0: trivial module)	m_{ij} arrows $V_i \to V_j$ if $V_j \otimes V = V_i^{\oplus m_{ij}} \oplus \sim$
Gabriel quiver	Corresponding to the set of non-isom, indecomp, projective modules of $\mathbb{C}[u,v]\#G$ $P_0, P_1, \dots, P_d$ $\quad (P_i = S_i \otimes \mathbb{C}[u,v])$	m_{ij} arrows $P_i \to P_j$ if multiplicity of P_i in $Q_{j,1}$ is m_{ij} where $0 \to Q_{j,n} \to \cdots \to Q_{j,1} \to P_j \to V_j \to 0$ min'l proj resolution of V_j as $\mathbb{C}[u,v]\#G$-modules
Resolution graph	corresponding to the set of irreducible components of the exceptional fiber for min'l resolution $\tilde{X} \to X$ $E_1, \dots, E_n$	have edge $E_i — E_j \iff E_i \cap E_j \neq \emptyset$ for $i \neq j$
AR-quiver (see Leuschke-Wiegand book Def 12.20)	corresponding to the set of indecomposable MCM modules $M_1, \dots, M_n$	r arrows $M_i \to M_j$ where $r = \dim \left(\mathrm{rad}(M_i, M_j) / \mathrm{rad}^2(M_i, M_j) \right)$ certain hom $\varphi: M_i \to M_j$ [add'l edges depending on "AR translates"]

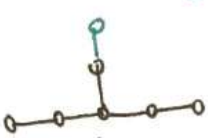
In classical setting,

all are ADE Dynkin diagrams

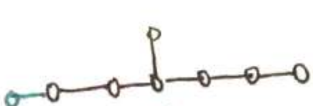
(or extended or doubled version)

$\tilde{A}_n$: 

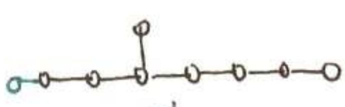
$\tilde{D}_n$: 



$\tilde{E}_6$



$\tilde{E}_7$

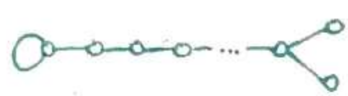


$\tilde{E}_8$

In noncommutative setting,

more Euclidean diagrams appear:

$\tilde{L}_n$: 

$\tilde{DL}_n$: 

Towards a noncommutative / quantum McKay correspondence

Replace:

$\mathbb{C}[u,v] \rightsquigarrow$ $\boxed{\text{A Artin-Schelter regular algebra of dim 2}}$

(def: connected graded $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots$, gldim 2, $\mathrm{Ext}^i_A(\mathbb{C}, A) = \begin{cases} A & i=2 \\ 0 & i \neq 0 \end{cases}$
of polynomial growth)

AS regular algebras are nice homological analogues of commutative poly'l algs, as discussed earlier in the week.

In gldim 2: $A \cong \underbrace{\mathbb{C}_q[u,v] := \mathbb{C}\langle u,v\rangle / (vu - quv)}_{q\text{-poly'l alg}}$, $q \in \mathbb{C}^*$ —OR— $\cong \mathbb{C}_J[u,v] := \mathbb{C}\langle u,v\rangle / (vu - uv - u^2)$

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finite group G $\rightsquigarrow$ Finite dim'l Hopf algebra

Hopf algebras are naturally viewed as a collection of symmetries of a quantum object/algebra

Classical Geom

action $\underset{\text{alg. grp}}{G} \curvearrowright \underset{\text{variety} \ldots\ \text{(some geom. object)}}{X}$

Commutative Alg

coaction of $\mathcal{O}(G)$ on $\mathcal{O}(X)$

$\mathcal{O}(G)$: comm. Hopf alg; $\mathcal{O}(X)$: comm alg

Noncom Geom

action $\underset{\text{quantum group}}{G_q} \curvearrowright \underset{\text{quantum object}}{X_q}$

Noncom Alg

coaction of $\mathcal{O}(G_q)$ on $\mathcal{O}(X_q)$

$\mathcal{O}(G_q)$: noncom Hopf alg; $\mathcal{O}(X_q)$: noncom alg

has a semisimple categories of modules (used to define McKay quiver)

use $\boxed{\text{semisimple Hopf algebras } H}$

$\Downarrow$
(finite dim'l)

$G \leq SL_2(\mathbb{C})$ that acts faithfully & linearly on $\mathbb{C}[u,v]$ $\rightsquigarrow$ $\underset{ss}{H} \curvearrowright \underset{\text{AS dim 2}}{A}$ so that A is an H-module algebra where

* H-action preserves grading of A
* H-action doesn't factor through smaller Hopf alg ("inner faithfulness")
* "hom'l determinant" of H-action on A is trivial

On homological determinant of H-action on AS-regular algebra A of dim d

Take: $\eta: H \to k$ so that H-action on $H^d_{\mathfrak{m}}(A)^*$ is given by $h \cdot \xi = \eta(h)\xi$

[the d^{th} local cohomology module wrt $\mathfrak{m} = A_1 \oplus A_2 \oplus \cdots$ max'l graded ideal of A
$= R^d\Gamma_{\mathfrak{m}}(A)$;
$\Gamma_{\mathfrak{m}}(A) = \{x \in A : A_{\geq n} x = 0$ for some $n \geq 1\}$
it is 1-dim'l; choose basis ξ

Take $S: H \to H$ antipode of H

Then $\operatorname{hdet}_H A = \eta \circ S : H \to k$; it is trivial if equal to counit map of H

Ex. $H = kG$, V faithful rep of G, $A = S(V)$
Then $\operatorname{hdet}_{kG} S(V) = \det_G : kG \to k$ (determinant map $G \to k$ extended linearly to kG)

Theorem (CKWZ, 2013) The pairs (H, A) so that $H \curvearrowright A$ satisfying the hypotheses above
* AS reg dim 2
are classified as follows:

Case	A (AS reg dim 2)	H (semisimple Hopf algebra)
classical case (a)	$\mathbb{C}[u,v]$	$\mathbb{C}G$, $G \leq SL_2(\mathbb{C})$ finite
(b)	$\mathbb{C}_{-1}[u,v] = \mathbb{C}\langle x,y\rangle/(yx+xy)$	$\mathbb{C}C_n$ $n \geq 2$ (diagonal action)
(c)		$\mathbb{C}C_2$ (non-diagonal action, $G = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$)
(d)		$\mathbb{C}D_{2n}$ for $n \geq 3$
(e)		$(\mathbb{C}D_{2n})^\circ$ for $n \geq 3$
(f)		$\mathcal{O}(G)^\circ$, $\mathcal{O}_{-1}(SL_2(\mathbb{C})) \twoheadrightarrow \mathcal{O}(G)$ [Bichon-Natale] fin-dim'l $G \leq SL_2(\mathbb{C})$ non-abelian, finite
(g)	$\mathbb{C}_q[u,v]$ $q^2 \neq 1$	$\mathbb{C}C_n$ $n \geq 2$ (diagonal action)
(h)	$\mathbb{C}_J[u,v]$	$\mathbb{C}C_2$ (diagonal action)

Noncom. McKay correspondence (CKWZ)

Two papers I. (case by case)
II. (adaptation of classical arguments)

Semisimple Hopf algebras H that act on AS reg algs A of dimension 2 linearly, not factoring, with trivial homological determinant

$(A, H) \longmapsto A^H$

Coordinate rings A^H of "quantum Kleinian singularities"

[I] Have that A^H is not AS regular

$\cong C/_{C\Omega}$, C AS reg dim 3, Ω normal elt in C

[I] McKay quiver (see next page)

NOT a bijection

resolution graph — open question ? not sure at the moment of the best way to desingularize...

Euclidean Diagrams

[II] H-grmod $\sim$ $A\#H$ - graded proj mod. (using same maps as classical case)

[II] Gabriel quiver

AR quiver ? unexplored at the moment

(graded) PROJECTIVE MODULES over $A\#H$

[I] Auslander's Theorem holds

$A\#H \cong End(A_{A^H})$

(graded) PROJECTIVE MODULES over $End(A_{A^H})$

categ. equiv

MCM MODULES over A^H

[II] NC. Herzog Thm

[II]

$\text{prmod-}End(A_{A^H}) \xleftarrow{\sim} add_{A^H} A$

[similar to before]

Conj: Auslander's Theorem holds for $H \curvearrowright A$ of higher dimension

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Additional Open Directions —

connection to: SING. CATEGs ? PREPROJ ALGS ? MATRIX FACTs ? CLUSTER THY ?

Upshot: As expected — homological generalization works out well for actions on AS reg. algs

Some details

McKay quiver:

vertices = { non-isomorphic simple left H-modules $V_0, V_1, \dots, V_d$ }

arrows: $V_i \xrightarrow[\vdots]{m_{ij}} V_j \iff V_j \otimes V = \sum_i m_{ij} V_i$

Take $V = A_1$ where $A = \mathbb{C} \oplus A_1 \oplus A_2 \oplus \cdots$

Case Euclidean Diagram	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
	$\tilde{A}-\tilde{D}-\tilde{E}$	$\tilde{A}_{n-1}$	$\tilde{L}_1$	$D_{\frac{n+4}{2}}$ n even; $D_{\frac{n+1}{2}}$ n odd	$\tilde{A}_{2n-1}$	$\tilde{D}-\tilde{E}$	$\tilde{A}_{n-1}$	$\tilde{A}_1$

$\hookrightarrow$ have that different pairs (A, H) correspond to same diagram

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Gabriel quiver (defined similarly to classical setting

Pf of Auslander's Theorem (case-by-case)

Case (a): ✓ (Auslander)

Cases (b)(g)(h) [diagonal action of cyclic groups] done by Mori–Ueyama

Cases (c)(d)(e)(f) use result of Bao–He–Zhang that
$$A \# H \cong \mathrm{End}(A_{A^H}) \iff (A\#H)/(1 \# t) \text{ is finite dimensional} \quad (*)$$
(t = integral of H)

* (c), (d) establish Right hand side of $(*)$ by brute force

* (e) also by brute force but with different computations than cases (c), (d)

* (f) Have that $\mathcal{O}(G)^{\Omega} = \mathbb{C}G$ for some twist Ω

* $G = D_{2m}$ & LHS of $(*)$ holds for (A, H) $\iff$ it holds for (A_{Ω}, H^{Ω})
— reduces to case (d)

* G of type E: establishes right hand side of $(*)$ by showing that $A^H \cong \mathbb{C}[u,v]^G$ via a careful analysis of McKay quiver...